

# A short review of *Defending the Axioms*

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The monograph *Defending the axioms: on the philosophical foundations of set theory* (2011) of Penelope Maddy, forms the third and more or less conclusive step in her decades of efforts towards a naturalistic characterization of and justification for the proper methods in early and contemporary set theory, of which the first and second tidemarks are her *Naturalism in mathematics* (1997) and *Second Philosophy: A Naturalistic Method* (2007) respectively. A brief sketch of answers to the addressed methodological question bypassing traditional philosophical issues of truth and existence is already contained in the 1997 book, and the goal of the current book, after the return to consideration of issues of truth and existence in the 2007 book, is to fill in and develop it. Although two apparently conflicting philosophical stands: Thin Realism and Arealism are elucidated in this monograph, neither are recommended by Maddy, who instead suggests the question of which is correct has less bite than it might appear. In the end, she seems hoping to base the defence for the proper methods on a type of mathematical objectivity independent of those troublesome philosophical issues of truth and existence, and suggesting that the only way to appreciate this objectivity is to dive into the active practice in set theory and mathematics in general.

The book consists of five chapters. Chapter 1 lays out the problem of defending the set theoretic axioms by a historical survey of how pure mathematics arose out of applied mathematics and how set theory originated and developed from there. Three key points are demonstrated in this survey: firstly, many mathematical studies, such as abstract group theory, arose as a result of mathematicians pursuit of a range of peculiarly mathematical goals with no immediate connection to applications(p.27); secondly, Euclidean geometry was no longer regarded as the true theory of physical space but merely the study of one among many abstract mathematical spaces; thirdly and finally, mathematical accounts of physical phenomenon are now not taken as literal truths but free-standing abstract models that resemble the world in complex ways.

Chapter 2 addresses the question: what are the proper methods for set theory? But before touching on this, Maddy firstly introduces the so called Second Philosophy from which she proposes to approach the above issue. As a rough approximation, a Second Philosopher is born native to empirical inquiry, from ordinary perception to more delicate hypothesis-experiment methods, trusting them while wiling to improve them from within, but accepting no demand to

justify her methods from outside. With this meta-philosophy in mind, Maddy then examines several cases from set-theoretic practice: Cantors and Dedekinds introduction of sets, Zermelos defence of his axiomatization, and the case for determinacy. And she concludes that the proper methods of set theory are just those the practice provides us, i.e. embracing effective means toward desired mathematical ends(p.52), which range from relatively local problem-solving, to providing foundations, to more open-ended pursuit of promising mathematical avenues(p.52), and argues that these methods, which are apparently different from the observation and experiment ones of natural science, are totally autonomous, not only independent of philosophical considerations, but also accepting no supplementation or correction from natural science, where the Second Philosopher starts his inquiry in the first place. However, while the Second Philosopher has no right to nag on the mathematical goals that mathematicians pursue and the means they use to achieve them, she still can ask whether the whole practice as a salient human activity lead us to truths about some subject matter, from outside of mathematics and standing within empirical science. And even assume sets do exist and set theory is a body of truths about them, the Second Philosopher may still wonder why the methods described above can make this, just like she could inquire the reasons why natural science provide us knowledge about the physical world. Maddy then briefly outlines Goedels well-known Platonistic metaphysics as an example trying to respond to these, and calls this general type of positions Robust Realism. But even ignoring the Benacerrafs Problem that has kept puzzling Robust Realism, the basic idea of the position that the goal of set theory is to describe an independently-existing reality conflicts with the proper methods of set theory, Maddy argues. To see this, it is sufficient to point out that the mere fact sets or an axiom about them serve this or that particular mathematical goal cannot guarantee the existence of them or the axiom holds in the universe of sets: the reality can be just sadly uncooperative(p.58). It is here that Maddy welcomes Thin Realism, which is intended to answer about the nature of set-theoretic activity in a way that both respects the actual methods of set theory and preserves the legitimacy of the pursuit of new axioms and a solution to the Continuum Problem.

Thin realism, expounded in Chapter 3, simply asserts that sets just are the sort of thing set theory describes. It does attribute some negative properties such as acausal, non-spatiotemporal, etc., to sets like Robust Realism does, since set theory tells us nothing about sets being in space or time, or participating in causal interactions, but would not go beyond the positive things asserted by set theory as Robust Realism would do, and in this sense it is thin. While Robust Realism requires a non-trivial account of the reliability of set-theoretic methods, Thin Realism sees it as a plain fact about what sets are(p.63). Taking the Continuum Problem as an example, Thin Realism uses the law of excluded middle, which is included by the proper methods of set theory, to show that CH has an objective and determinate truth value, while Robust Realism wants a full-bodied metaphysical theory of what sets are that will ratify CH in a more substantive way(p.64). To make Thin realism clear, Maddy draw some contrasts of it to three more positions, related to Kant, Carnap and Wright respectively,

other than Robust realism, but I am not going to cover them in this review, together with many other topics in Chapter 3.

In Chapter 4, Maddy presents an apparently rival stand of Thin Realism: Arealism. Recall that Thin realism begins its story with the assumption that set theory is a body of truth. Arealism, in contrast, regards set theory and pure mathematics not as a body of truth at all but rather a spectacularly successful enterprise with its own methods, assuming that the great role mathematics has played in natural science does not constitute a proof for its truth, although Arealism does think this key role distinguishes mathematics from other human endeavours which may also have their proper methods different from those usual to the Second Philosopher, e.g. astrology, theology. Given this characterization, one might take Arealism as a kind of Nominalism or Fictionalism or Formalism, etc., but as Maddy emphasizes, these names could be misleading, for instance, Arealism does not entertain with any preconceptions about what knowledge must be like that would seem to rule out knowledge of sets, as Nominalism is likely to do, and similar subtleties exist for the cases of Fictionalism and Formalism. When it comes to the relation between Arealism and Thin Realism, Maddy points out, despite their disagreement over truth and existence, the two positions are indistinguishable at the level of method, and much deeper, the objective facts that underlie the two positions are exactly the same, namely, the topography of mathematical depth(p.100). To make a final choice between them, Maddy returns to the problem that has been set aside: does the Second Philosopher have good reason to judge set theory as a body of truth? The answer is no, since the indispensability argument is rejected by the Second Philosopher(cf. Maddy 1997), and no other good reasons can be found by her. But there is no conclusive evidence for set theory not being a body of truth either, Maddy insists. So the final proposal is: Thin Realism and Arealism are equally accurate, second philosophical descriptions of the nature of pure mathematics(p.112).

In Chapter 5, also the last one, Maddy tries to draw some morals from the above conclusion. The most central one is that we should shift our attention away from the questions of truth and existence to the challenge of understanding the common objectivity, the mathematical depth, underlying both Thin Realism and Arealism. As a beginning to appreciate this objectivity, Maddy, after a revisit to Robust Realism, returns to the original problem of defending the axioms by considering more examples from set-theoretic practice and the interrelations of intrinsic and extrinsic justifications. And the last sentence of the monograph is: What does matter, all that really matters, is the fruitfulness and promise of the mathematics itself, which highlights the point of the whole book.

While I share the conviction of naturalism with Maddy, I do have some different ideas about Thin Realism and Arealism. The conclusion that they are equally accurate cannot satisfy me, and I think it is quite natural for the Second Philosopher to ask whether any acausal non-spatiotemporal things like sets exist or not, and finding out the answer is not something that does not matter for her understanding of the world. Actually, I think she would fairly

choose Arealism rather than the other, since viewing set theory as a human practice and human beings as bio-chemical systems living in a physical world, as the Second Philosopher would agree, it is absurd to regard the practice is related to anything beyond this physical world. Here Maddy might argue that the Second Philosopher should not have any preconception of what knowledge must be like, but why not? The Second Philosopher studies human beings ,and their cognitive capacities particularly, in empirical science, surely she has good reason to judge on what they can know and what they cannot, just like she can judge on what they can eat and digest and what they cannot. So if the Second Philosopher has no good reason for taking set theory as a body of truth, as Maddy would agree, why does not she adopt the position of Arealism and done and done. Furthermore, Thin Realism as described in the book seems self-contradictory to me. On the one hand, it takes sets as existing objectively rather than constituted by set-theoretic methods; on the other hand, it does not allow the sets to be uncooperative with the methods.As I see it, to keep Realism thin is to change it into Arealism.

However, what is really important and intriguing in the book is not the odd attitude to Thin Realism and Arealism, but the emphasis of the mathematical depth that actually drives mathematics. I appreciate this very much, recommending it to be attended by all philosophers of mathematics. And as in her many other works, Maddy manifests a deep and wide knowledge about the mathematical practice, especially that in contemporary set theory, which makes the book quite informative.